Fall 2015 Math 245 Exam 1 Solutions

Problem 1. Carefully define each of the following terms:

a. contrapositive

The **contrapositive** of conditional proposition $p \to q$ is $(\sim q) \to (\sim p)$.

b. valid

An argument/proof is **valid** if the conclusion must be true if all the premises are true.

c. tautology

A (compound) proposition is a **tautology** if it is true regardless of the truth values of any other (constituent) propositions.

d. vacuous proof

A **vacuous proof** proves the implication $p \to q$ by proving that $\sim p$ holds.

e. proof by contradiction

A **proof by contradiction** takes as additional hypothesis the negation of the desired conclusion, and derives a contradiction.

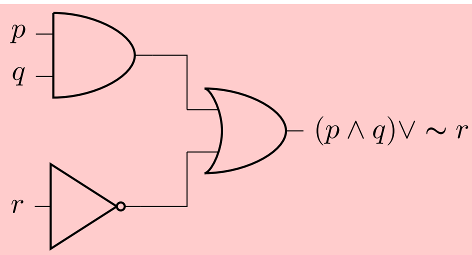
Problem 2. Define the terms "proposition" and "predicate", and explain the difference.

A **proposition** is a statement that must be true or false, but not both or neither. A **predicate** is a collection of propositions indexed by one or more variables. A predicate is not a proposition, because it may be true for certain values of its variable(s) and false for others.

Problem 3. Write the negation of the proposition $\forall x \in \mathbb{R}, \exists y \in \mathbb{Z}, \forall z \in \mathbb{R},\ x > zy$, and simplify your result to eliminate $\sim$.

$\sim \forall x \in \mathbb{R}, \exists y \in \mathbb{Z}, \forall z \in \mathbb{R},\ x > zy \quad \equiv \quad \exists x \in \mathbb{R}, \forall y \in \mathbb{Z}, \exists z \in \mathbb{R},\ x \leq zy.$

Problem 4. Construct the circuit corresponding to the Boolean expression $(p \wedge q) \vee \sim r$.



Problem 5. Write the converse of the inverse of the contrapositive of $p \to (q \vee r)$.

Contrapositive: $\sim (q \vee r) \to \sim p$. Inverse of the contrapositive: $(q \vee r) \to p$.
Converse of the inverse of the contrapositive: $p \to (q \vee r)$.

Problem 6. Use a truth table to determine whether $(p \oplus q) \vee r \equiv p \oplus (q \vee r)$.

p	q	r	$p \oplus q$	$(p \oplus q) \vee r$	$q \vee r$	$p \oplus (q \vee r)$
T	T	T	F	$\boxed{\text{T}}$	T	$\boxed{\text{F}}$
T	T	F	F	F	T	F
T	F	T	T	$\boxed{\text{T}}$	T	$\boxed{\text{F}}$
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

The two propositions are *not* equivalent, because in two of the eight rows, the truth values disagree in the fifth and seventh columns (circled).

Problem 7. Disprove the following statement: $\forall x \in \mathbb{R}$, if $x > 0$ then $\frac{1}{x+2} = \frac{1}{x} + \frac{1}{2}$.

We need a counterexample, some <u>specific</u> $x \in \mathbb{R}$ such that $x > 0$ and $\frac{1}{x+2} \neq \frac{1}{x} + \frac{1}{2}$. Fortunately we don't need to look far, as every choice we might make will work. For example, $x = 1$ has $1 > 0$ and $\frac{1}{1+2} = \frac{1}{3} \neq 1.5 = \frac{1}{1} + \frac{1}{2}$.

Problem 8. Fill in the missing justifications, including line numbers, for the following proof.

1.	$(p \vee q) \rightarrow r$	hypothesis
2.	$\sim q \rightarrow c$	hypothesis
3.	p	hypothesis
4.	q	Rule of contradiction on 2.
5.	$p \vee q$	Disjunctive addition on 4.
6.	r	Modus ponens on 1,5.
7.	$\therefore p \wedge r$	Conjunctive addition on 3,6.

Problem 9. Carefully state the definition of $\lceil x \rceil$, and find some $y \in \mathbb{R}$ with $\lceil y \rceil > y^2$.

For $x \in \mathbb{R}$, we define $\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$. Alternatively, in words, $\lceil x \rceil$ is the smallest integer that is greater than or equal to the real number x. It's a bit tricky to find y, as those $y \leq 0, y = 1$, or $y \geq \sqrt{2}$ all fail to satisfy the desired condition. However all other y work. For example, take $y = 0.5$. We have $\lceil y \rceil = 1 > 0.25 = y^2$.

Problem 10. Use mathematical induction to prove that, for all natural $n \geq 2$,

$$2 + 3 + \cdots + n = \frac{(n-1)(n+2)}{2}.$$

Base case: $n = 2$. The left hand side has one summand, 2, and the right hand side is $\frac{(2-1)(2+2)}{2} = 2$.
Alternative base case: We can actually use as base case $n = 1$, in which case the LHS has no summands, so is 0, while the right hand side is $\frac{(1-1)(2+2)}{2} = 0$.
Inductive case: Assume as inductive hypothesis that $2 + 3 + \cdots + n = \frac{(n-1)(n+2)}{2}$. Add $(n+1)$ to both sides, getting $2 + 3 + \cdots + n + (n+1) = \frac{(n-1)(n+2)}{2} + \frac{2n+2}{2} = \frac{n^2+3n}{2} = \frac{n(n+3)}{2}$.
Alternative inductive case: We may instead assume $2 + 3 + \cdots + (n-1) = \frac{(n-2)(n+1)}{2}$ and add n to both sides, which after algebra gives $2 + 3 + \cdots + n = \frac{(n-1)(n+2)}{2}$.